NAKE Workshop
June 15, 2001
Numerical Methods for Dynamic Economic Analysis
Ken Judd

Bifurcation Methods: an Asset Market Example

Asymptotics

- Sometimes IFT fails
- There exists extensions of IFT which can be used
  - Bifurcation methods – multiple branches
  - Newton's polygon method – fractional powers

Asset Market Analysis – Example of Bifurcation

- Use simple single-period model
- Use Taylor-style expansions to solve for equilibrium of economies with small risk.
  - Disadvantage: results are proven valid only for cases with small risk
  - Advantage: focus on moments, estimable statistics
  - Advantage: more reliable than alternative approaches which are never valid.
  - Conjecture: validity is much greater than the proofs indicate
  - Disadvantage: complex algebraic manipulations
  - Solution: Use computer algebra software
- Results
  - Completely solves equilibrium for a collection of economies
  - Computes the impact on welfare of new assets
  - Computes optimal new asset

Demand with Two Assets

- Simple problem which displays general mathematical difficulty
- Two assets:
  - Safe asset: pay \$1 today, receive R tomorrow
  - Risky asset: pay \$1 today, receive $Z = R + \epsilon z + \epsilon^2 \pi$ tomorrow
  - All proceeds consumed tomorrow
- First-order condition for θ:

$$\begin{aligned} 0 &= E\{u'(R + \theta(\epsilon z + \epsilon^2 \pi))\,(z + \epsilon\pi)\} \\ &\equiv G(\theta, \epsilon) \end{aligned}$$

- Demand function:

$$G(\theta(\epsilon), \epsilon) = 0$$

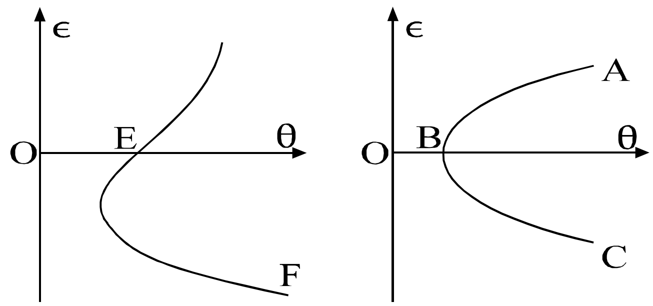
- Want to find expansion

$$\theta(\epsilon) = \theta_0 + \theta'(0)\epsilon + \frac{1}{2}\theta''(0)\epsilon^2 + ...$$

- Question: What is θ_0?
- Problem: θ is indeterminate at $\epsilon = 0$!

Bifurcation Approach

- Problem: $G(\theta, 0) = 0$ for all θ.
  - Demand at $\epsilon = 0$ is not well-defined.
  - However, it is defined at all $\epsilon \neq 0$
- B and E are bifurcation points



Bifurcation possibilities for asset demand

- Want to find expansion

$$\theta(\epsilon) = \theta_0 + \theta'(0)\epsilon + \frac{1}{2}\theta''(0)\epsilon^2 + ...$$

- Problem 1: Need to find

$$\theta_0 \equiv \lim_{\epsilon \to 0} \theta(\epsilon)$$

- Problem 2: Implicit differentiation of $G(\theta(\epsilon), \epsilon) = 0$ with respect to ϵ implies

$$\begin{aligned} 0 &= G_\theta(\theta(\epsilon), \epsilon)\theta' + G_\epsilon(\theta(\epsilon), \epsilon) \\ \Rightarrow & \ \theta'(0) = -\frac{G_\epsilon(\theta_0, 0)}{G_\theta(\theta_0, 0)} \end{aligned}$$

  but $G_\theta(\theta_0, 0) = 0$ because $G(\theta, 0) = 0$ for all θ!

- Solution to both problems:
  - $\theta'(0)$ is well-defined only if $G_\epsilon(\theta(0), 0) = 0$.
  - Hence, $G_\epsilon(\theta_0, 0) = 0$ must hold and fixes θ_0.
  - Given θ_0, L'Hospital's rule implies

$$\begin{aligned} \theta'(0) &= -\frac{G_\epsilon(\theta_0, 0)}{G_\theta(\theta_0, 0)} \\ &= -\frac{G_{\epsilon\epsilon}(\theta_0, 0)}{G_{\theta\epsilon}(\theta_0, 0)} \end{aligned}$$

    which is well-defined if $G_{\theta\epsilon}(\theta_0, 0) \neq 0$.

- In asset demand problem

$$\begin{aligned} G_\epsilon(\theta,0) &= E\left\{u''(R)\theta z^2\right\} + u'(R)\pi = 0 \\ &\implies \theta_0 = -\frac{u'(R)}{u''(R)}\,\frac{\pi}{\sigma_z^2} = \tau\,\frac{\pi}{\sigma_z^2} \end{aligned}$$

- Bifurcation theorem makes validates this procedure - a generalization of L'Hospital's rule.

- Simple asymptotic portfolio rule: θ_0 is product of

  – risk tolerance (a utility parameter)

$$\tau = -\frac{u'(R)}{u''(R)}$$

  – and the price of risk (a statistic of risk)

$$\frac{\pi}{\sigma_z^2}$$

Linear Approximation

- Calculate $\theta'(0)$ via implicit differentiation:

$$\begin{aligned} 0 &= G_{\theta\theta}\,\theta'\theta' + 2G_{\theta\epsilon}\,\theta' + G_\theta\theta'' + G_{\epsilon\epsilon} \\ &= 0\ \cdot\ \theta'\theta' + 2G_{\theta\epsilon}\,\theta' + 0\ \cdot\ \theta'' + G_{\epsilon\epsilon} \\ &\Longrightarrow\ \theta'(0) = -G_{\theta\epsilon}^{-1}G_{\epsilon\epsilon} \end{aligned}$$

- Solvability again depends on $G_{\theta\epsilon}(\theta_0, 0) \neq 0$
- In asset case $G_{\theta\epsilon}(\theta_0, 0) \neq 0$ and

$$\theta'(0) = -\ \frac{1}{2}\,\frac{u'''(R)}{u''(R)}\frac{E\{z^3\}}{\sigma_z^2}\,\theta_0^2$$

- Relative form (a.k.a. $\widehat{\theta}$) is

$$\begin{aligned} \frac{\theta'(0)}{\theta_0} &= \rho\,\frac{\pi}{\sigma_z^2}\,\frac{E\{z^3\}}{\sigma_z^2} \\ \rho &= \frac{1}{2}\frac{u'}{u''}\frac{u'''}{u''}(R) \end{aligned}$$

  - Depends on third-order properties of utility; we call ρ skew tolerance
  - Depends on third moment of return

- Linear approximation is

$$\begin{aligned} \theta(\epsilon) &\doteq \theta_0 + \theta'(0)\,\epsilon \\ &\doteq -\ \frac{u'(R)}{u''(R)}\,\frac{\pi}{\sigma_z^2} - \frac{1}{2}\,\frac{u'''(R)}{u''(R)}\frac{E\{z^3\}}{\sigma_z^2}\,\left(\frac{u'(R)}{u''(R)}\,\frac{\pi}{\sigma_z^2}\right)^2\,\epsilon \end{aligned}$$

Higher-order Approximations

- We compute higher-order approximations via implicit differentiation
  - We solve a succession of linear equations, just as in Implicit Function Theorem.
  - Solvability at each stage follows from $G_{\theta\epsilon}(\theta_0, 0) \neq 0$
  - Necessary assumptions:
    * Differentiability of u
    * Finite moments for z
- Second-order approximation is

$$\frac{\theta''(0)}{\theta_0} = \left((6\rho - 2)\, E\{z^2\} \;+\; 4\rho^2 \frac{E\{z^3\}^2}{E\{z^2\}^2} \;+\; \kappa \frac{E\{z^4\}}{E\{z^2\}} \right) \left(\frac{\pi}{\sigma_z^2} \right)^2 \tag{1}$$

where kurtosis tolerance at c is

$$\kappa = -\frac{1}{3} \frac{u'}{u''} \frac{u'}{u''} \frac{u''''}{u''}$$

Asset Market Equilibrium with Small Risks

- Safe asset yields R per dollar invested
- Examine a continuum of economies
- Risky asset in ϵ-economy yields $R(1+\epsilon z)$ per share
- Price for risky asset in ϵ-economy is

$$p(\epsilon) = 1 + \epsilon E\{z\} + \epsilon^2 \pi(\epsilon).$$

- Type i endowment: a_i dollars and θ_i^e shares
- Final wealth and consumption is

$$Y_i = \theta_i R(1+\epsilon z) + B_i R.$$

- Market clearing condition:

$$\theta_1 + \theta_2 = 1.$$

- First order condition for θ_i:

$$E\{u_i^{'}(Y_i)(\epsilon z + \epsilon^2 \pi)\} = 0.$$

- Equilibrium is (let $\theta_1 = \theta$, $\theta_2 = 1 - \theta$)

$$0 = E\{u_i^{'}(Y_i)(z + \epsilon\pi\},\ i = 1, 2$$

- For each ϵ we have two equations and two unknowns, θ and π.
- At $\epsilon = 0$ all (θ, π) values are equilibria.

- Bifurcation structure applies
  - At $\epsilon = 0$ asset prices are fixed at 1, but π and θ are indeterminate.
  - At $\epsilon \neq 0$ asset prices and allocation are determinate.

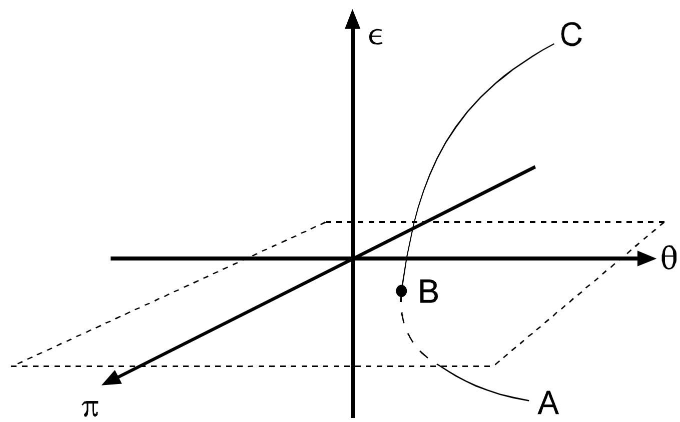


Figure 1:

General Formulas

- Direct computation (assume $E\{z\} = 0$) yields bifurcation point

$$\begin{aligned} \theta_0 &= \frac{u_1'/u_1''}{u_2'/u_2'' + u_1'/u_1''} = \frac{\tau_1}{\tau_1 + \tau_2} \\ \pi_0 &= -\frac{RE\left\{z^2\right\}}{\tau_1 + \tau_2} \end{aligned}$$

- Direct computation yields linear approximation for small ϵ :

$$\begin{aligned} \theta'(0) &= R\,\frac{\tau_1}{\tau_1+\tau_2}\,\frac{\tau_2}{\tau_1+\tau_2}\,\frac{\rho_1-\rho_2}{\tau_1+\tau_2}\,\frac{E\{z^3\}}{\sigma_z^2} \\ \pi'(0) &= \frac{2R^2}{(\tau_1+\tau_2)^2}\left(\frac{\tau_1}{\tau_1+\tau_2}\rho_1 + \frac{\tau_2}{\tau_1+\tau_2}\rho_2\right)\,\frac{E\{z^3\}}{\sigma_z^2} \end{aligned}$$

  - Change in θ governed by
    - differences in risk aversion and skew tolerance
    - skewness variance ratio
  - Risk premium ($-\pi$) decreases with positive skewness and u''', $\rho > 0$

- Direct computation produces arbitrary order approximation as long as u has sufficient derivatives and moments are finite.

- Examples indicate that low-order expansions are quite good approximations for sensible cases: efficient computation.

Comparing Market Structures

- Expected utility, prices, trade, etc. are all functions of ϵ
- For each asset configuration m, we can compute expected utility as a function of ϵ

$$U_{i,m}(\epsilon) = U_{i,m}(0) + U'_{i,m}(0)\epsilon + \frac{1}{2}U''_{i,m}(0)\epsilon^2 + \ldots.$$

- To compare markets m and n, we compute the difference in expansions

$$\Delta_{i,m,n}(\epsilon) = U_{i,m}(\epsilon) - U_{i,n}(\epsilon)$$

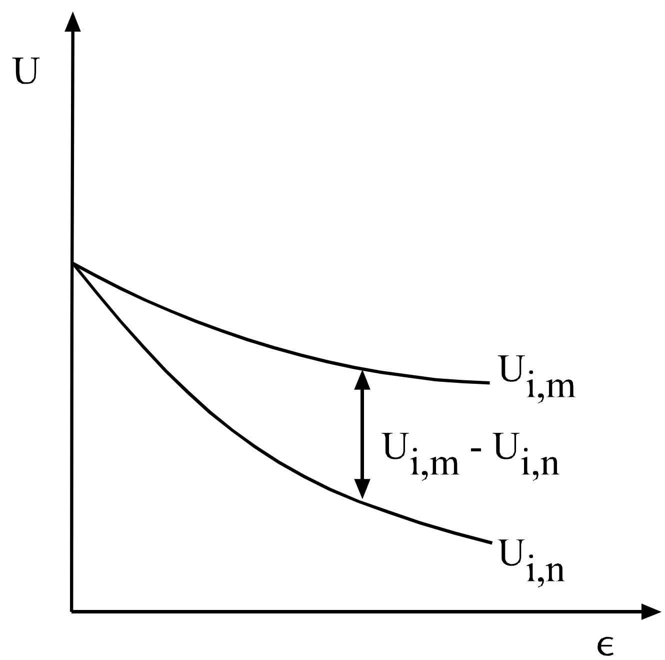


Figure 2: Difference in Asset Market Structures

- Can compute similar expressions for prices, trading volume, etc.

Adding a Derivative Asset

- Suppose we add a new asset, $y = f(z)$; agent 1 buys ϕ units.
- WLOG, $y = \overline{y} + \alpha z + \nu$ where $0 = E\{\nu\} = E\{z\nu\}$

Trading Patterns for the New Asset

- $\phi(\epsilon)$ is trader 1 purchases
- Bifurcation calculation shows $\phi(0) = 0$.
- If $\phi'(0) > 0$, then trader 1 buys and trader 2 sells the new asset.
- Asymptotic results revolve around two indices:
  - Difference in skew tolerance, $\rho_1 - \rho_2$,(utility parameters)
  - Covariance between z^2 and ν, $Cov(\nu, z^2)$

Theorem 1 *Type 1 agents buy y if ν is positively correlated with the tails of z and type 1 skew tolerance exceeds type 2 skew tolerance:*

$$\phi'(0) \;=\; R\,\frac{\tau_1\tau_2\left(\rho_1-\rho_2\right)}{\left(\tau_1+\tau_2\right)^3}\,Cov(\nu, z^2)$$

$$\phi'(0) > 0 \;\; iff \;\; (\rho_1 - \rho_2)\;\; Cov(\nu, z^2) > 0$$

Asset Holding Effects of the New Asset

Theorem 2 *Let $\theta_z^b(\epsilon)$ and $\theta_z^a(\epsilon)$ denote type 1 investors' holding of old asset before and after the new asset being introduced. Then*

$$\theta_z^a(\epsilon) - \theta_z^b(\epsilon) = -R\frac{\tau_1\tau_2\left(\rho_1 - \rho_2\right)}{\left(\tau_1 + \tau_2\right)^3}\alpha Cov(\nu, z^2)\frac{\epsilon^2}{2} + O(\epsilon^3).$$

Note that

$$\theta_z^a(\epsilon) - \theta_z^b(\epsilon) = -\ \alpha\ \phi'(0)$$

- If $\alpha > 0$, change in equity holdings is opposite of purchases of new asset.
- If $\alpha = 0$, no change in equity holdings to order $O(\epsilon^2)$.

Price Effects of the New Asset

Theorem 3 *Let $P_z^b(\epsilon)$ and $P_z^a(\epsilon)$ denote the equilibrium price of the first risky asset before and after the new asset being introduced.*

$$P_z^a(\epsilon) - P_z^b(\epsilon) = 8R^3 \frac{\tau_1 \tau_2 \left(\rho_1 - \rho_2\right)^2}{\left(\tau_1 + \tau_2\right)^5} \frac{E\left\{\nu z^2\right\}^2}{E\left\{z^2\right\}} \frac{\epsilon^4}{2} + O(\epsilon^5) > 0$$

In particular, the initial risky asset rises in value and rises more as the new asset is more correlated to the tails of the original asset.

- Robust result: *any* nontrivial new asset will increase the price of the old asset.
- Price change depends on third-order properties of the utility function.
- Price change depends on $E\left\{\nu z^2\right\}$
- No third- (or first- or second-) order impact if $E\left\{\nu z^2\right\} = 0$.

Welfare Effects of the New Asset

Theorem 4 *The asymptotic welfare change from adding the asset y for trader 1, measured by the unit-free consumption equivalent, equals*

$$\frac{U_1^a(\epsilon) - U_1^b(\epsilon)}{u_1^{'}} = \frac{R^4 \tau_1^2 \tau_2^2 (\rho_1 - \rho_2)^2}{2(\tau_1 + \tau_2)^5} \times \left(5\left(\frac{\theta_1^e}{\tau_1} - \frac{\theta_2^e}{\tau_2}\right) + \theta_2^e\right) \frac{E\left\{\nu z^2\right\}^2}{E\left\{z^2\right\}} \epsilon^4 + O(\epsilon^5)$$

where θ_i^e is i's endowment of stock. Similarly for the second trader's welfare change, $[U_2^a(\epsilon) - U_2^b(\epsilon)]/u_2^{'}$.

- Welfare effect on individuals is ambiguous
  - Pareto-improving if tastes and endowment are not too dissimilar
    - No difference in skew-tolerance, i.e., $\rho_1 - \rho_2 = 0$, then no welfare change to $O(\epsilon^4)$.
    - Volume (and type 1's purchases) is proportional to

$$\frac{\theta_2^e}{\tau_2} - \frac{\theta_1^e}{\tau_1}$$

  - Loss is possible
    - If type 1 buys much equity then he loses from price increase.
    - Someone must gain – single good case.
  - Skew-tolerance difference magnifies welfare change from other factors.
  - Co-skewness of new asset with old, $E\left\{\nu z^2\right\}$, is in dominant term

- Asymptotic unanimity:
  - If some new asset makes all better off, then any new asset does.
  - If agent i prefers y' to y, then so does $j \neq i$.
- Results depend solely on utility function properties and moments of asset returns, not on number of states.
- Social welfare, measured in wealth terms, changes by the positive quantity

$$\begin{aligned} \Delta SW &= \frac{U_1^a(\epsilon) - U_1^b(\epsilon)}{u_1^{'}} + \frac{U_2^a(\epsilon) - U_2^b(\epsilon)}{u_2^{'}} \\ &= \frac{R^4 \tau_1^2 \tau_2^2 \left(\rho_1 - \rho_2\right)^2}{2\left(\tau_1 + \tau_2\right)^5} \frac{E\left\{\nu z^2\right\}^2}{E\left\{z^2\right\}} \epsilon^4 + O(\epsilon^5) \end{aligned} \quad (2)$$

- Computational notes
  - All derivations were done using Mathematica
  - Intermediate terms in utility expansion numbered over 10,000.
  - If we began with specific utility functions and return distributions, then computational effort would be much smaller, competitive with general methods.

Optimal New Asset: General Approach

- Few results based on special cases, often assuming
  - *Assumed* traded assets were linear combinations of factors, or
  - *Assumed* a new asset is a linear combination of factors
- We make no assumption about returns or utility, and compute asymptotically optimal new asset
- Social Welfare
  - For new asset y and ϵ, type i equilibrium utility is $U_i(y,\epsilon)$.
  - Any social welfare function is, for some weight $\alpha > 0$,

$$W(y,\epsilon) = \alpha U_1(y,\epsilon) + (1-\alpha)U_2(y,\epsilon)$$

  - The optimal new asset problem is

$$\max_y W(y,\epsilon) \tag{3}$$

- Let $Y(\epsilon)$ be the optimal new asset for the ϵ economy.
  - The function $Y(\epsilon)$ is defined by the solution to the first-order condition

$$0 = W_y(Y(\epsilon),\epsilon)$$

  - At $\epsilon = 0$ all new assets are redundant, implying $0 = W_{yy}(y,0)$
- Apply bifurcation theorem to find $Y(0)$. Either
  - We want a new asset and $W_{yy\epsilon}(Y(\epsilon),\epsilon)$ is negative definite, or
  - It is optimal to introduce no new asset.

Optimal New Asset

- Find $y = f(z)$ which maximizes

$$(E\left\{yz^2\right\})^2 = Cov\left(y, z^2\right)^2 \tag{4}$$

which is sufficient since the other terms in the change in utility are unrelated to y.

- Normalizations we assumed above
  - Scale y so that $\sigma_y^2 = 1$.
  - Uncorrelated to bonds: $E\{y\} = 0$
  - Uncorrelated to stocks: $E\{yz\} = 0$.

- Optimal asset solves a (degenerate) calculus of variations problem.:

$$\begin{array}{ll} \max_f & \left(\int f(z)z^2 d\mu(z)\right)^2 \\ s.t. & \int f(z)d\mu(z) = 0 \\ & \int zf(z)d\mu(z) = 0 \\ & \int f(z)^2 d\mu(z) = 1, \end{array} \tag{5}$$

Theorem 5 *As riskiness goes to zero, any optimal new asset is asymptotically equivalent up to order* ϵ^4 *to adding* $y = z^2$.

- Optimal new asset (or, collection thereof) is independent of tastes: Asymptotic unanimity
- Optimal second option to introduce is z^3

Theorem 6 *As riskiness goes to zero, any pair of optimal new assets is asymptotically equivalent up to order* ϵ^5 *to adding* $y = z^2$ *and* $y = z^3$.

- Conjecture: "Complete the moments to complete the market"

Theorem 7 *In two-type model, one optimal asset achieves the Pareto frontier. As riskiness goes to zero,*

1. *up to order* ϵ^4*, it is asymptotically equivalent to adding* $y = z^2$*,*
2. *up to order* ϵ^5*, it is asymptotically equivalent to adding* $y = z^2$ *and* $y = z^3$*,*
3. *up to order* ϵ^5*, it is asymptotically equivalent to adding some linear combination of* $y = z^2$ *and* $y = z^3$*, and*
4. *these equivalences are independent of tastes and returns.*

Alternative Approaches

- Samuelson (1970) took a polynomial approximation approach, replacing utility function with polynomial approximation, and solve nonlinear equations involving utility function derivatives and return moments.
- Magill (1977): derived linear approximations for dynamic programming problems with small additive shocks
- Kydland-Prescott, McGrattan, and King-Plosser-Rebelo offer *ad hoc* approaches
  - Idea: replace objective and/or first-order conditions and budget constraint with linear (or loglinear) approximations to arrive at an analytically tractable set of equations.
  - Magill (1977) shows general invalidity of this approach
  - This approach is at best a certainty equivalent and first-order
  - Judd (1996, 1998) show that McGrattan method can produce nonsensical results
  - Tesar (1995) uses KP-KPR approach and shows that first welfare theorem is false(?!)
  - Kim-Kim (1999) demonstrates general invalidity of KP-KPR approach; offers a new *ad hoc* approach since "macroeconomists will not use available mathematically valid approaches."
- Similar Campbell (1993) procedure is also *ad hoc*, not based on IFT or extensions
- Mathematics and Mathematical Economics Literature
  - Fleming (1971) and Fleming-Souganides (1986) examine similar problems.
  - See Judd (1996) for several citations.
- Bifurcation and related IFTs produce theorems as well as high-quality numerical approximations.

Other Applications of IFT and Bifurcation Methods

- Arbitrary riskiness, similar utility
- Dynamic trading in options not spanned by stocks and bonds (Leisen-Judd)
- Asymmetric information, close to symmetric information
- Multiple periods in a recursive economy

Conclusions

- Bifurcation methods are natural extensions of IFT for asset market problems
- Bifurcation approach avoids *ad hoc* assumptions usually used for sake of tractability
- Bifurcation approach produces asymptotically valid results, avoiding *ad hoc* approximation schemes.
- Small noise results can be extended via computations in non-small cases
- Results depend on natural conditions on statistics and tastes
- Bifurcation methods have substantial computational value

The Role of Computation in Economic Analysis

- Traditional roles
  - Empirical analysis
  - Applied general equilibrium
- Nontraditional roles
  - Substitute for theory
  - Complement for theory
- Questions:
  - What can computational methods do?
  - Where does computation fit into economic methodology?

Computation and Science

- The Scientific Method
  - Experimentation – detect patterns
  - Theories, models, and deductive methods – produce theorems, closed-form solutions
  - Computations
- Computational Successes in Science
  - The Red Spot of Jupiter
  - Origin of the moon
  - Shape of Galaxies
- Computation in Science and Economics
  - Astronomy and economics – observational sciences
  - Red Spot $\sim$ Kydland-Prescott RBC success
  - Common challenge: "Visualization" problems
  - Differences: Precise theories of science versus qualitative theories in economics

What can we compute now?

- Optimization
  - Dynamic programming
  - Mechanism design
- General equilibrium
  - Arrow-Debreu general equilibrium
  - General equilibrium with incomplete markets
  - General equilibrium with imperfect competition
  - Dynamic, perfect foresight models
  - Dynamic, stochastic recursive models
- Asset markets
  - Asymmetric information - Grossman-Stiglitz, Radner
  - Imperfect competition - Kyle model
- Games
  - Finite games- Lemke-Howson, Wilson, McKelvey
  - Supergames - Cronshaw-Luenberger, Judd-Yeltekin-Conklin
- Dynamic games
  - Closed-loop (a.k.a., Markov perfect) - Kotlikoff-Shoven, Wright-Williams, Miranda-Rui, Vedenov-Miranda, Sibert
  - Supergames with states - Judd-Yeltekin

Progress in Numerical Analysis

- Linear programming - Interior point methods
- Nonlinear equations, complementarity problems
- Only small amount of numerical analysis is used in economics

Software Progress

- Parallelism: Combine many cheap processors
- Program development tools

Hardware Progress

- Moore's law for semiconductors
- Optical computing
- DNA computing
- Quantum computing

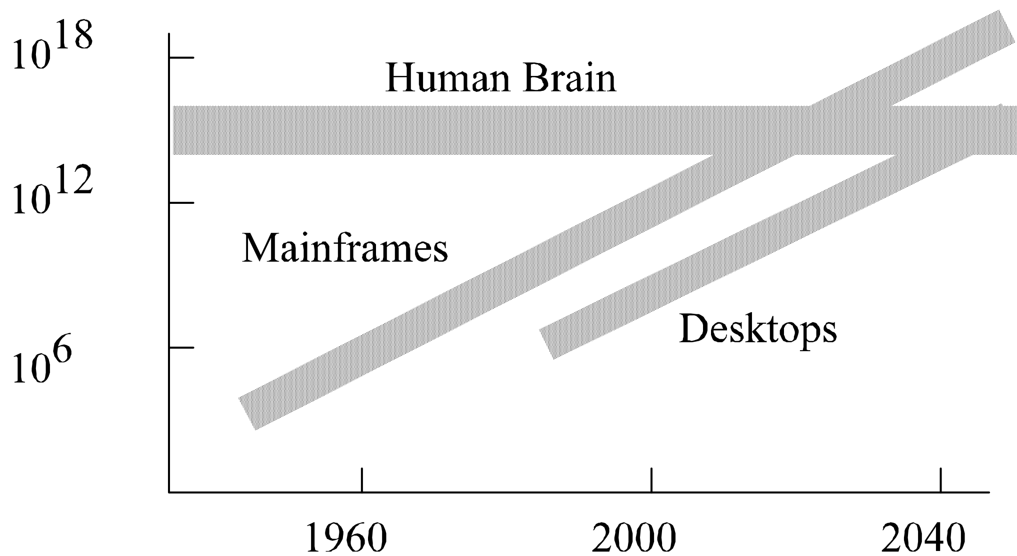


Figure 3: Trends in computation speed: flops vs. year

Modes of Theoretical Analysis

- Theory: A Definition
  - Define terms, concepts
  - State assumptions
  - Determine the implications of the theory
- Two ways to ascertain implications of a theory
  - Deductive Theory
    - Prove general theorems about general case
    - Add auxiliary assumptions to make tractable
    - Prove more precise theorems about tractable cases
  - Computational Theory
    - Specify parameterized versions of the theory
    - Compute specific models, i.e., fix parameter values
    - Summarize results of computations
- Examples
  - I.O. – oligopoly, theory of firm, info.
  - Labor – principal-agent, compensation
  - Finance – market microstructure, info.
  - Poli Sci – legislative, election models

- Theory as Exploration: The Typical Scenario
  - A theory defines a general class of models
  - Deductive theory can examine, without error, a thread containing a continuum of instances of the theory
  - Computation can examine, with error, a finite collection of general instances
  - The properties of the model are piecewise continuous in the parameters

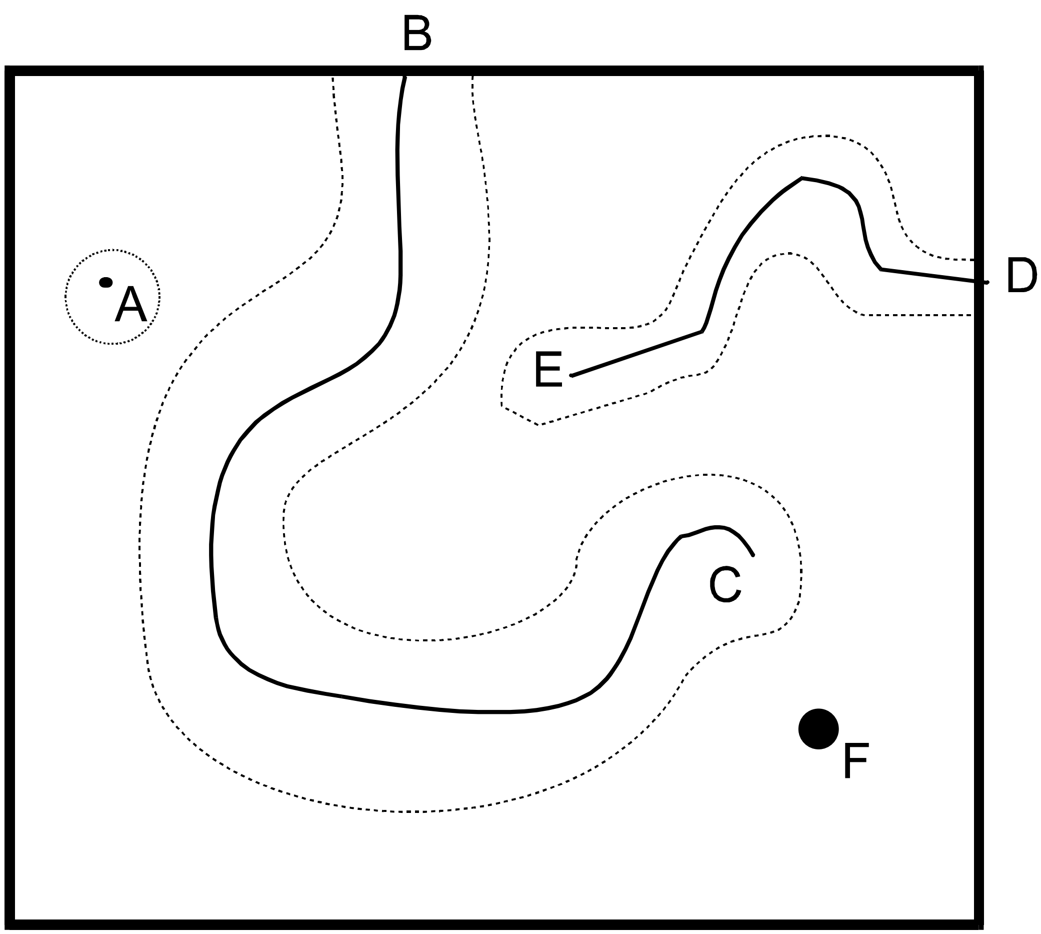


Figure 4: Typical graph of tractable cases

- Theoretical Practice: Science vs. Economics
  - Professional rewards: E.g., Einstein credited with general relativity, but he never proved a general existence theorem, produced no nontrivial solution.
  - Approximate "computational" methods pervade physics
  - Ad hoc mathematics often used in theoretical physics
  - Economic theorists follow "Bourbaki", pure math
- Computational versus Deductive Theory
  - Deductive theory produces absolute truths
  - Deductive methods can build a "deep" theory, as in mathematics
  - Deductive theory focusses on a narrow collection of excessively simple models
  - Is mathematics, Bourbaki-style, appropriate for economics?

Three Examples of Computational Theory

- Haubrich
  - Question: Executive rewards
  - Model: principal-agent model
  - Observations: executives get only about $3 per thousand
  - Conventional wisdom: executives should get more for incentives - supported by risk-neutral agent specification
  - Computations: compute optimal contract for reasonable tastes and technology
  - Results: optimal share is often about $3 per thousand
- Spear-Srivastava and Phelan-Townsend
  - Computation and theory as "Tag Team" partners
  - A theory begins with an enormous collection of possible results: payments in repeated moral hazard problems can depend on entire history
  - Theory can narrow range of possibilities: S-S reduced problem to 1-D dynamic programming problem
  - Computation uses theoretical analysis to construct efficient computational methods: P-T papers

- Quirmbach
  - Question: What ex post market structure best encourages ex ante innovation among competitors?
  - Model: A two-period model of innovation then production
  - Computations: Search across demand functions, costs, R&D success rates, game forms
  - Results: Bertrand and Cournot were roughly the same, better than ex post collusion.
  - There are no reasonable theorems.

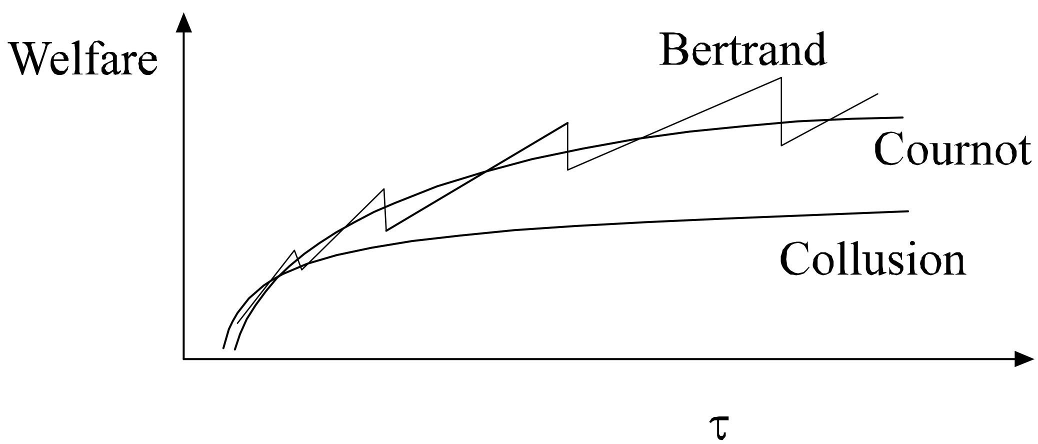


Figure 5: Welfare vs. probability of innovation success

Systematic Approaches to Computational Theory

- Perturbation Methods: Fattening the Thread
  - $F(x, \delta, \epsilon) = 0$ expresses a theory, with general solution $x(\delta, \epsilon)$
  - Deductive theory solves $F(x, \delta, 0) = 0$ to get $x(\delta, 0)$ – the thread of special cases.
  - Perturbation methods compute $x(\delta, \epsilon)$ for small ϵ – fattens the thread.
  - Example: consumption function in growth models

$$C(k, \sigma^2) \doteq C(k^*, 0) + C_k(k^*, 0) + C_{\sigma^2}(k^*, 0)\sigma^2 + \cdots$$

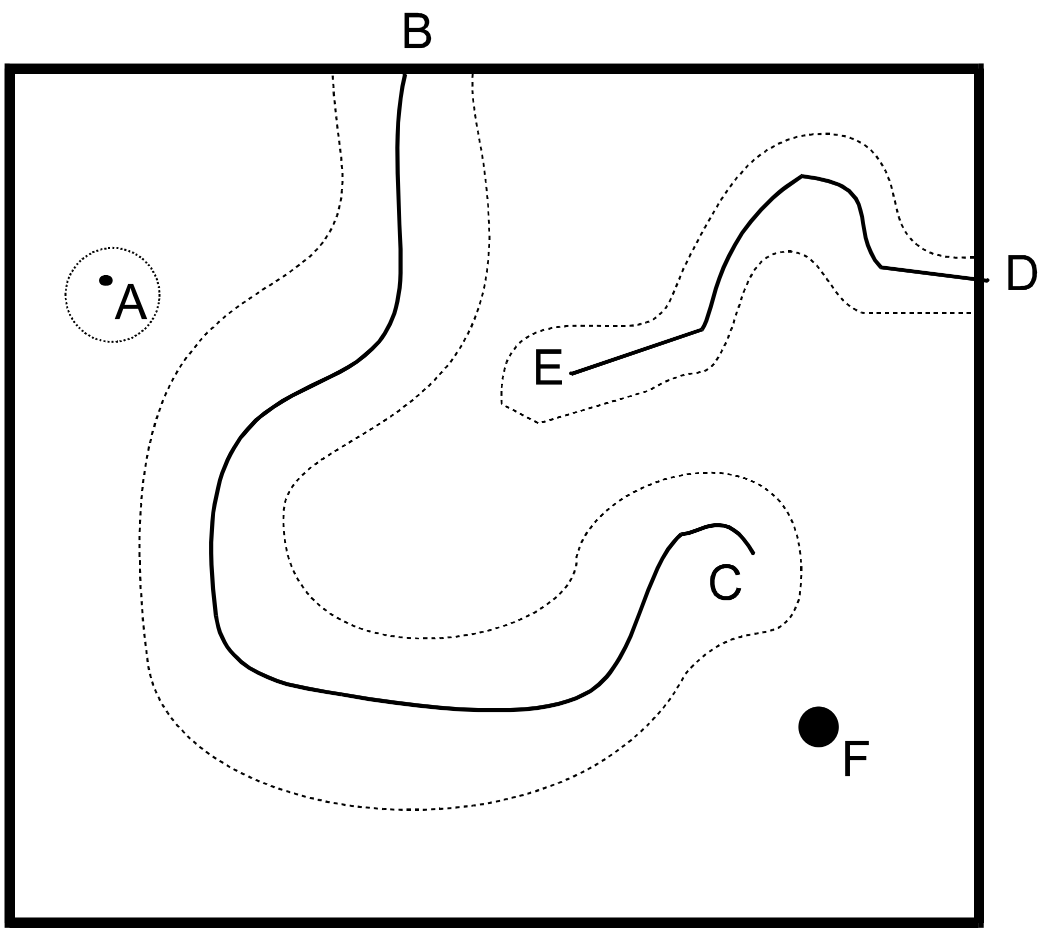


Figure 6: Typical graph of tractable cases

- Monte Carlo Sampling
  - Draw N points independently from the model space according to a probability measure μ.
  - Test proposition at each sampled instance
  - If proposition is true at each sampled instance, then state "We conclude with $1-(1-\epsilon)^N$ confidence that the set of counterexamples has μ-measure less than ϵ."

- Quasi-Monte Carlo Sampling
  - Construct a N-point set with low discrepancy (i.e., "uniformly spread") in a metric d. Let mesh be δ.
  - Test proposition at each sampled instance
  - If proposition is true at each sampled instance, then "No set of counterexamples contains a ball of diameter δ."
  - If proposition is true at each sampled instance, interval arithmetic is used, and Lipschitz bounds apply and can be computed a priori, then Proposition is *proven*!

- Regression Methods of Summarizing Results
  - Construct a N-point set of instances.
  - Compute quantities of interest (price, quantity, welfare, etd.) at each sampled instance
  - "Regress" the quantities of interest on the model parameters, compute "covariances" among model parameters and equilibrium outcomes.

- Presentation of Computational Results
  - Tables, Graphs – limited
  - "Confidence probabilities"
  - "Regression" results
  - Need ways to describe robust patterns

Computational Theory vs. Calibration

- Calibration
  - Specify a model
  - Consult empirical facts to choose a case
  - Analyze a single case
  - Inconsistent with Bayesian decision theory
- Computational theory
  - Only loosely constrained by data
  - Look for patterns - comparative statics

Deductive versus Computational Theory

	Deductive methods	Computational methods
Approach:	Prove theorems	compute examples
Validity:	absolute	limited by numerical error
Range:	continua	finite number of examples
Generality:	simplifications made for tractability	limited only by computational methods
Existence:	proven	present examples of ϵ-equilibrium
Efficiency:	(dis)proven	indicate quantitative importance
Comp. statics:	usually need special functional forms	impose empirically motivated restrictions
Errors:	specification errors	numerical errors
Inputs:	mathematical theory skills	computer time, computational skills

- Synergies
  - Numerical examples inform deductive analyses
  - Computational simplifications from deductive theory
- Tradeoffs and trends
  - Error type
    * Specification errors of deductive theory - trend?
    * Numerical errors of computational methods - falling rapidly
  - Input Costs
    * Human math skills and knowledge - some trend
    * Computation costs ($/Flop) - falling rapidly

The Future of Computational Economics

- Technology – Hardware and Software
  - Computing costs will continue to decrease
  - New computing environments and technologies can be exploited
  - Economists will catch up to numerical analysis frontier
  - Numerical analysis will develop better methods to exploit new technologies
  - Economists will develop of problem-specific methods (as in CGE)
- An Economic Theory of the Future
  - Inputs: Human time and computers
  - Outputs: Understanding of economic systems
  - Trend: Falling price of computation
  - Prediction: Comparative advantage principles imply
    * Substitution of computer power for human time and effort in analysis of specific models
    * Human activity will specialize on formulating theoretical concepts and models, and deciding which problems are most important.