NAKE Workshop
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Numerical Methods for Dynamic Economic Analysis
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Regular Perturbation Methods

Taylor Series

Suppose that $h : R^n \to R^m$.

- Linear approximation at $x = x_0$ is

$$h(x) \doteq h(x_0) + h_x(x_0)(x - x_0)$$

- Taylor series approximation of $h(x)$ based at x_0

$$h(x) \doteq h(x_0) + h_x(x_0)(x - x_0) + (x - x_0)^\top h_{xx}(x_0)(x - x_0) + ...$$

  and computed via repeated differentiation.

- A function h and its Taylor series at x_0 have equal low-order derivatives at x_0.
- Taylor series is a numerical approximation technique, useful when h and its derivatives are easily computed at some special point x_0.
- We exploit the formal theory of analytic functions: A function $h : R^n \to R$ is *analytic* at x iff h equals a power series on some open neighborhood of x.

Implicit Function Theorem

- Suppose $h : R^n \to R^m$ is defined implicitly in $H(x, h(x)) = 0$ where $H : R^n \times R^m \to R^m$.
  - Suppose $H(x_0, y_0) = 0$; then $h(x_0) = y_0$.
  - Implicit differentiation shows

    $$H_x(x, h(x)) + H_y(x, h(x))h_x(x) = 0$$

    implying

    $$h_x(x_0) = -H_y(x_0, y_0)^{-1} H_x(x_0, y_0)$$

    if $H_y(x_0, y_0)$ is nonsingular.
- Linear approximation for $h(x)$
  - Formula

    $$h^L(x) \doteq h(x_0) + h_x(x_0)(x - x_0)$$

  - Quality check:
    - Compute

      $$E = H(y, h^L(y))$$

    - If $E < \epsilon$, then we have an ϵ-equilibrium.

- Compute tigher-order terms by repeated differentiation.
  - Differentiate again

    $$H_{xx} + 2H_{xy}h_x + H_{yy}h_xh_x + H_yh_{xx} = 0$$

  - This defines $h_{xx}(x_0)$ since H_y is assumed nonsingular
  - Construct the quadratic approximation

    $$h^Q(x) \doteq h(x_0) + h_x(x_0)(x - x_0) + \frac{1}{2}(x - x_0)^\top h_{xx}(x_0)(x - x_0)$$

  - Check its quality by computing $E = H(x, h^Q(x))$.

General Regular Perturbation Strategy

- Find special case where one knows solution
- Use implicit function theorem ideas to compute nearby cases
- Check to see if solution is good for problem of interest
- Hardware and software
  - Human brains and hands
    - Slow
    - Unreliable
  - Computers and symbolic software (Mathematica, Maple, Reduce, Mathcad, Macsyma)
    - Reliable
    - Tireless
    - Fast

Deterministic Growth

Consider the dynamic programming problem

$$\begin{aligned} V(k_0) &\equiv \max_{c(t)} \int_0^\infty e^{-\rho t} u(c) dt \\ \dot{k} &= f(k) - c \\ k(0) &= k_0 \end{aligned}$$

- Bellman equation:

$$\rho V(k) = \max_c \; u(c) + V'(k)(f(k) - c) \tag{1}$$

- $C(k)$: policy function defined by

$$\begin{aligned} 0 &= u'(C(k)) - V'(k) && (2) \\ \rho V'(k) &= V''(k)(f(k) - C(k)) + V'(k) f'(k) && (1_k) \end{aligned}$$

- Steady State:

$$\begin{aligned} \rho &= f'(k^*) \\ C(k^*) &= f(k^*) \end{aligned}$$

- Want Taylor expansion:

$$\begin{aligned} C(k) &\doteq C(k^*) + C'(k^*)(k - k^*) + C''(k^*)(k - k^*)^2/2 + ... \\ V(k) &\doteq V(k^*) + V'(k^*)(k - k^*) + V''(k^*)(k - k^*)^2/2 + ... \end{aligned}$$

- We have: $k^*, \; V(k^*), \; C(k^*), \; f'(k^*), V'(k^*)$.

Linearize around a steady state:

- Differentiate (1_k,2) w.r.t. k:

$$\begin{aligned}\rho V'' &= V'''(f-C) + V''(f'-C') + V''f' + V'f'' && (1_{kk}) \\ 0 &= u''C' - V'' && (2_k)\end{aligned}$$

- At the steady state

$$0 = -V''(k^*)C'(k^*) + V''(k^*)f'(k^*) + V'(k^*)f''(k^*) \qquad (1_k^*)$$

  - Substituting (2_k) into (1_k^*) yields

    $$0 = -u''(C')^2 + u''C'f' + V'f''$$

    with two solutions

    $$C'(k^*) = \frac{\rho}{2}\left(1 \pm \sqrt{1 + \frac{4u'(C(k^*))f''(k^*)}{u''(C'(k^*))f'(k^*)f'(k^*)}}\right)$$

  - However, we know $C'(k^*) > 0$; hence, take positive solution

Higher-Order Expansions

- Conventional perception in macroeconomics: "perturbation methods of order higher than one are considerably more complicated than the traditional linear-quadratic case ..." – Marcet (1994, p. 111)
- Mathematics Literature: See Bensoussan and many others
- Compute $C''(k^*)$ and $V'''(k^*)$.
  - Differentiate $(1_{kk}, 2_k)$:

$$\begin{aligned} \rho V''' &= V''''(f-C) + 2V'''(f'-C') + V''(f''-C'') \qquad (1_{kkk}) \\ &\quad + V'''f' + 2V''f'' + V'f''' \\ 0 &= u'''(C')^2 + u''C'' - V''' \qquad (2_{kk}) \end{aligned}$$

  - At k^*, (1_{kkk}) reduces to

$$0 = 2V'''(f'-C') + 3V''f'' - V''C'' + V'f''' \qquad (1^*_{kkk})$$

- The only unknowns in $(1^*_{kkk}, 2^*_{kk})$ are $C''(k^*)$ and $V'''(k^*)$.
- The equations $(1^*_{kkk}, 2^*_{kk})$ are *LINEAR* in the unknowns.
- General terms
  - The linear system for order n is

$$\begin{pmatrix} u'' & -1 \\ V'' & -n(f'-C') \end{pmatrix} \begin{pmatrix} C^{(n)} \\ V^{(n+1)} \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

    for some known terms A_1 and A_2.
  - Higher-order terms are produced by solving linear systems
  - The linear system is always determinate since $-nu''(f'-C') + V''$, the determinant, is always negative.

Accuracy Measure

Consider the one-period relative Euler equation error:

$$E(k) = 1 - \frac{V'(k)}{u'(C(k))}$$

- Equilibrium requires it to be zero.
- $E(k)$ is measure of optimization error
  - 1 is unacceptably large
  - Values such as .00001 is a limit for people.
  - $E(k)$ is unit-free.
- Define the L^p, $1 \leq p < \infty$, *bounded rationality accuracy* to be

$$\log_{10} \parallel E(k) \parallel_p$$

- The L^∞ error is the maximum value of $E(k)$.

Global Quality of Asymptotic Approximations

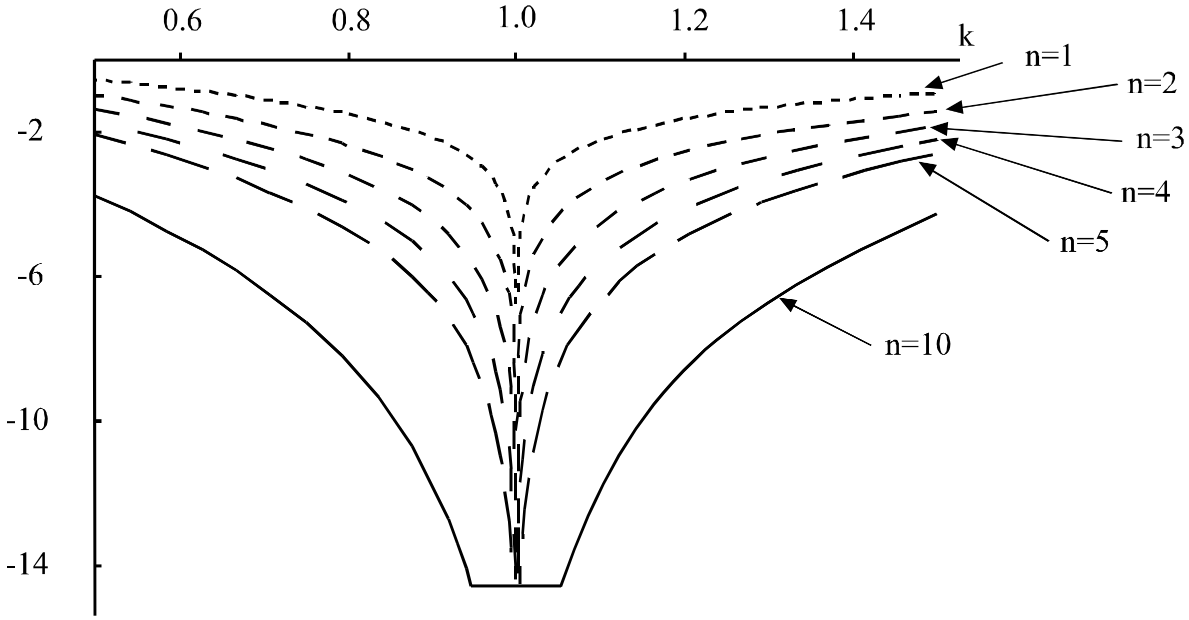

Figure 1: Graph of $\log_{10} E(k)$

Magill (1977) Alternative

- Replace nonlinear problem with similar linear quadratic problem.
- Produces correct linear approximation when done correctly.
- Frequently used in RBC analysis, such as Kydland-Prescott (1982).
- Not extendable to higher-order approximations.

Comparison with McGrattan (1990)

- True Problem

$$\begin{aligned} V(k) &\equiv \max_{c(t)} \int_0^\infty e^{-\rho t} u(c) dt \\ \dot{k} &= f(k) - c \end{aligned}$$

- McGrattan: "Replace objective with quadratic approximation and constraint with linear approximation, both at the deterministic steady state"

$$\begin{aligned} &\max_{c(t)} \int_0^\infty e^{-\rho t} (u(c^*) + u'(c^*)(c - c^*) + u'(c^*)(c - c^*)^2/2) dt \\ \dot{k} &= f(k^*) + f'(k)(k - k^*) - c = f(k^*) + \rho(k - k^*) - c \end{aligned}$$

- Result is

$$C^{McG}(k) = f(k^*) + \rho(k - k^*) \neq C(k^*) + C'(k^*)(k - k^*)$$

- Lesson: Casual approximations are not reliable. Only formal, perturbation calculations should be used.

Single-Sector, Stochastic Growth

Consider

$$\begin{aligned} \max_{c} \quad & E\left\{\int_0^\infty e^{-\rho t} u(c)dt\right\} \\ & dk = (f(k) - c)dt + \frac{1}{2}\sqrt{\sigma}kdz \end{aligned}$$

- Define the value function (making dependence on σ explicit)

$$V(k, \sigma) = \sup_{c \in \mathcal{F}} E\left\{\int_0^\infty e^{-\rho t} u(c)dt\right\}$$

- Bellman equation:

$$\rho V(k, \sigma) = \max_{c}\; u(c) + V_k(k)\; (f(k) - c) + \sigma k^2 V_{kk}(k, \sigma)$$

- Taylor expansions of C and V:

$$\begin{aligned} C(k, \sigma) \;\doteq\; & C(k^*, 0) \\ & + C_k(k^*, 0)(k - k^*) + C_\sigma(k^*, 0)\sigma \\ & + C_{kk}(k^*, 0)(k - k^*)^2/2 + C_{\sigma k}(k^*, 0)\sigma(k - k^*) \\ & \quad + C_{\sigma\sigma}(k^*, 0)\sigma^2/2 + ... \end{aligned}$$

$$\begin{aligned} V(k, \sigma) \;\doteq\; & V(k^*, 0) \\ & + V_k(k^*, 0)(k - k^*) + V_\sigma(k^*, 0)\sigma \\ & + V_{kk}(k^*, 0)(k - k^*)^2/2 + V_{\sigma k}(k^*, 0)\sigma(k - k^*) \\ & \quad + V_{\sigma\sigma}(k^*, 0)\sigma^2/2 + ... \end{aligned}$$

- Solution defined by two equations:
  - Bellman equation

$$0 = -\rho V + u(C) + V_k(f - C) + \sigma k^2 V_{kk} \tag{1}$$

  - First-order condition

$$0 = u'(C) - V_k \tag{2}$$

- Derivatives $V(k, 0)$ are known from deterministic problem
- Compute $V_\sigma(k^*, 0)$
  - Differentiating the system (1,2) with respect to σ

$$0 = -\rho V_\sigma + V_{k\sigma}(f - C) + V_{kk}k^2 + \sigma V_{kk\sigma}k^2 \tag{1_σ}$$

$$0 = u'' C_\sigma - V_{k\sigma} \tag{2_σ}$$

    - At $k = k^*$ and $\sigma = 0$, we know that $C = f$.
    - Furthermore, $u' = V_k$ always.
    - Therefore, at $k = k^*$ and $\sigma = 0$, (1_σ) reduces to

$$V_\sigma(k^*) = V_{kk}(k^*)k^2\rho^{-1} \tag{1_σ^*}$$

  - V_σ proportional to
    - output variance, $\sigma[k^*]^2$
    - $V_{kk}(k^*)$, the curvature of the value function for the deterministic problem at its steady state.

- Similarly compute C_σ and $V_{k\sigma}$, etc.

$$V_{k\sigma} = \frac{V_{kkk}k^2 + 2kV_{kk}}{C_k}$$

$$C_\sigma = \frac{V_{kkk}k^2 + 2kV_{kk}}{C_k u''}$$

- Note:
  - Only linear equations arise
  - *Even if u is quadratic and f linear,* $C_\sigma \neq 0$!
  - C_σ and $V_{k\sigma}$ are functions only of the third derivative of the value function and the steady state value of consumption in the deterministic case, which we computed in the previous section.
  - We could continue this for higher order expansions.

Multidimensional Stochastic Optimal Control

Consider

$$V(x, \sigma) \equiv \max_u \quad E\left\{ \int_0^\infty e^{-\rho t} \pi(x, u) dt \right\}$$
$$dx = f(x, u)dt + \sqrt{\sigma/2}\, dz$$

Objective:

Compute multivariate Taylor series expansion of $V(x, \sigma)$ and $U(x, \sigma)$

Procedure:

- Compute steady state of deterministic ($\sigma = 0$) case.
- Compute linear approximation of deterministic problem: a *quadratic matrix* problem.
- Compute multivariate Taylor series expansion of deterministic equations and solve for higher-order terms: *linear* problems.
- Compute multivariate Taylor series expansion of stochastic equations and solve for certainty nonequivalence correction terms in σ: *linear* problems.

Observations:

- Discrete-time versions are also possible, but messier.
- Limitations
  - Not global
  - Smoothness requirements
- Advantages
  - Fast
  - Results are theorems about local behavior
  - Computer algebra systems can do the hard work
- Conclusions
  - Higher-order approximations
  - Certainty non-equivalent approximations
  - Easy to compute
  - Far more accurate than linear approximations

Dynamic Games

- Procedure:
  - Parameterize a continuum of games where $\varepsilon = 0$ is a degenerate case.
  - Compute equilibrium of degenerate case (perhaps using perturbation)
  - Compute perturbation equations (via differentiation)
  - Solve (usually linear) equations for how equilibrium (value functions and strategy functions) change as ε increases away from zero.
- Three Examples:
  - Patent race:
    - Degenerate case: Prize is zero, equilibrium effort is zero
    - Perturbation parameter: Prize
  - Dynamic duopoly:
    - Degenerate case: no interaction (cross-elasticity in demand), equilibrium is two monopoly solutions
    - Perturbation parameter: interaction (non-zero cross-elasticity)
  - Learning Curve:
    - Degenerate case: no learning curve, equilibrium is repeated static solution
    - Perturbation parameter: parameters governing learning curve